On a Certain Difference-Differential Equation Related to Nonlinear Lattices (Non-Linear Waves : Classical Theory and Quantum Theory)

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On a certain difference-differential equation related to nonlinear lattices

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In this note I should like to point out that the solutions of equations of the form

$$\frac{\psi(u_1+v_1,\ u_2+v_2)\psi(u_1-v_1,\ u_2-v_2)-\psi(u_1,u_2)^2}{\psi(u_1,\ u_2)^2}$$

$$=\left(F_1^2(v_1)\frac{\partial^2}{\partial u_1^2}+2X(v_1,v_2)F_1(v_1)F_2(v_2)\frac{\partial^2}{\partial u_1\partial u_2}+F_2^2(v_2)\frac{\partial^2}{\partial u_2^2}\right)\log\psi(u_1,u_2) \tag{1}$$

are related to multi-soliton solutions of a discrete-time lattice equation.

<u>Example 1</u>. The function

$$\psi(u)=1+e^u$$

satisfies

$$\frac{\psi(u+v)\psi(u-v)-\psi(u)^2}{\psi(u)^2}=F(v)\frac{d^2}{du^2}\log\psi(u)$$

where

$$F^2(v)=4\sinh^2\frac{v}{2}$$

<u>Example 2</u>. The function

$$\psi(u_1,\ u_2)=1+e^{u_1}+e^{u_2}+Be^{u_1+u_2}$$

satisfies (1) with

$$F_1=2\sinh\frac{v_1}{2}\quad,\quad F_2=2\sinh\frac{v_2}{2}\quad(v_1,\ v_2\gtrless 0)$$

and

$$X=\frac{1}{2(B-1)\sinh\frac{v_1}{2}\sinh\frac{v_2}{2}}\left[B\sinh^2\frac{v_1+v_2}{2}+\sinh^2\frac{v_1-v_2}{2}-(B+1)\left\{\sinh^2\frac{v_1}{2}+\sinh^2\frac{v_2}{2}\right\}\right]$$

Application. Let n be lattice position and t the discrete-time, and

$$u_1 = 2\eta_1 = 2(P_1 n + \beta_1 t + \eta_1^0)$$
$$u_2 = 2\eta_2 = 2(P_2 n + \beta_2 t + \eta_2^0)$$

and let

$$\psi_n(t) = 1 + e^{2\eta_1} + e^{2\eta_2} + B e^{2\eta_1 + 2\eta_2} ,$$

We assume a discrete-time lattice equation

$$\delta^{-2}[\psi_n(t+\delta)\psi_n(t-\delta) - \psi_n(t)^2] = \psi_{n+1}(t)\psi_{n-1}(t) - \psi_n(t)^2 .$$

Then we see that β_i (i=1,2) and B are determined as

$$\delta^{-1} \sinh \beta_i \delta = \sinh P_i \qquad (i=1,2)$$

$$B = \frac{\sinh^2(P_1 - P_2) - \delta^{-2}\sinh(\beta_1 - \beta_2)\delta}{\delta^{-2}\sinh(\beta_1 + \beta_2)\delta - \sinh^2(P_1 + P_2)} ,$$

and ψ_n(t) gives a two-soliton solution. ψ_n(t) reduces to a solution for the ordinary exponential lattice in the limit as $t \to 0$.

Extension 1. As an extension of (1) we have

$$\frac{\psi(u_1+v_1, u_2+v_2, \dots)\psi(u_1-v_1, u_2-v_2, \dots)}{\psi(u_1, u_2, \dots)^2}$$
$$= 1 + \sum_{i,j=1}^{N} E_{ij}(v_1, v_2, \dots)\frac{\partial^2}{\partial u_i \partial u_j} \log \psi(u_1, u_2, \dots) .$$

Solutions to this equation will lead us to a N-soliton solution.

Extension 2. As is well known, the elliptic ϑ-function $\vartheta_3(u)$ satisfies an equation of the form

$$\frac{\vartheta_3(u+v)\vartheta_3(u-v)}{\vartheta_3(u)^2} = C(v) + A(v)\frac{d^2}{du^2}\log\vartheta_3(u)$$

and ϑ_3 is related to the simplest periodic solution of the exponential lattice. For the future problem to obtain general periodic solutions of the discrete-time lattice equation, we will have to deal with a certain equation of the form

$$\frac{\psi(u_1+v_1,\ u_2+v_2,\ \ldots)\,\psi(u_1-v_1,\ u_2-v_2,\ \ldots)}{\psi(u_1,\ u_2,\ \ \ )^2}$$

$$= C(v_1, v_2, \ldots) + \sum_{i,j} E_{ij}(v_1, v_2, \ \ )\frac{\partial^2}{\partial u_i \partial u_j}\log\psi(u_1, u_2, \ldots).$$

Reference

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